Math 146C - Ordinary and Partial Differential Equations III - Spring 2011
April 28, 2011
Practice Midterm

Name: _____

| Problem | Score |
|---|---|
| 1 | /25 |
| 2 | /25 |
| 3 | /25 |
| 4 | /25 |
| Score | /100 |

**Problem 1** (25 points)**.** *Find all the eigenvalues and eigenfunctions for the boundary value problem*

$$y'' + \lambda y = 0, \quad y'(0) = y(\pi) = 0.$$

*Answer.* The eigenvalues are

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2, \quad n \in \mathbb{N}$$

with eigenfunctions

$$y_n = \cos\left(\frac{(2n-1)t}{2}\right), \quad n \in \mathbb{N}.$$

□

**Problem 2** (25 points)**.** *Find the Fourier series for the function*

$$f(x) = \begin{cases} x, & -\pi \leq x < 0 \\ 0, & 0 \leq x < \pi \end{cases}$$

*assuming $f$ has period $2\pi$.*

*Answer.* The Fourier series is

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \left( \frac{2}{(2n-1)^2 \pi} \cos(2n-1)x + \frac{(-1)^{n+1}}{n} \sin nx \right).$$

□

**Problem 3** (25 points)**.** *Find the solution of the heat conduction problem*

$$\begin{cases} u_{xx} = u_t, \quad 0 < x < 1, \quad t > 0 \\ u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0 \\ u(x,0) = x^2 \end{cases}$$

*Answer.* The solution is

$$u(x,t) = \sum_{n\ odd}^{\infty} \left( \left( \frac{2(n\pi)^2 + 8}{(n\pi)^3} \right) e^{-n^2\pi^2 t} \sin n\pi x \right) + \sum_{n\ even}^{\infty} \left( \left( \frac{-2}{n\pi} \right) e^{-n^2\pi^2 t} \sin n\pi x \right).$$

□

**Problem 4** (25 points)**.** *Consider the dispersive wave equation*

$$u_{tt} + u = u_{xx}, \quad 0 < x < \pi, \quad t > 0$$

*with the boundary conditions*

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0$$

*and the initial conditions*

$$u(x,0) = f(x), \quad u_t(x,0) = 0, \quad 0 < x < \pi.$$

*Use the method of separation of variables to find a series solution to the above problem.*

*Answer.* The solution is

$$u(x,t) = \sum_{n=1}^{\infty} k_n \sin nx \cos t\sqrt{1+n^2}$$

where

$$k_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx.$$

□